JOURNAL OF APPROXIMATION THEORY **63**, 303–320 (1990)

Polyharmonic Cardinal Splines: A Minimization Property*

W. R. MADYCH

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06268

AND

S. A. NELSON

Department of Mathematics, Iowa State University, Ames, Iowa 50011

Communicated by E. W. Cheney

Received March 9, 1989; revised January 17, 1990

Polyharmonic cardinal splines are distributions which are annihilated by iterates of the Laplacian in the complement of a lattice in Euclidean n-space and satisfy certain continuity conditions. Some of the basic properties were recorded in our earlier paper on the subject. Here we show that such splines solve a variational problem analogous to the univariate case considered by I. J. Schoenberg.

1. INTRODUCTION

Recall that a k-harmonic cardinal spline is a tempered distribution u on R^n which satisfies

$$\begin{aligned} &\text{(i)} \quad u \text{ is in } C^{2k-n-1}(R^n) \\ &\text{(ii)} \quad \Delta^k u = 0 \qquad \text{on} \quad R^n \backslash Z^n. \end{aligned} \tag{1}$$

Here Δ is the usual Laplace operator defined by

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

and, if k is greater than one, Δ^k denotes its kth iterate, $\Delta^k u = \Delta(\Delta^{k-1} u)$. Of course $\Delta^1 = \Delta$ and Z^n denotes the lattice of points in R^n all of whose coordinates are integers.

* Partially supported by a grant from the Air Force Office of Scientific Research, AFOSR-86-0145.

0021-9045/90 $3.00

Such distributions were considered in [1] where their basic properties were recorded and our motivation for studying them was indicated. One of the key developments there was the existence and uniqueness of the solution to the so-called cardinal interpolation problem for k-harmonic splines. This result may be summarized as follows: Given a sequence $\{a_{\mathbf{j}}\}$, $\mathbf{j}$ in Z^n, of polynomial growth and an integer k satisfying $2k \geqslant n+1$ there is a unique k-harmonic spline f such that $f(\mathbf{j}) = a_{\mathbf{j}}$ for all $\mathbf{j}$ in Z^n.

In this paper we continue recording properties of these distributions. Specifically we show that under appropriate conditions the k-harmonic splines are solutions of a variational problem. These properties together with those considered in [1] are remarkably similar to well-known properties of the univariate cardinal splines of I. J. Schoenberg; see [2]. For example, much of the material in this paper parallels matter found in [2, Chap. 6]. On the other hand, because of the non-existence of B-splines with compact support in the general multivariate case, our development is significantly different from that found there.

The variational problem alluded to above is considered in the context of the space $L_k^2(R^n)$, the class of those tempered distributions whose derivatives of order k are square integrable. The properties of this class and its discrete analogue which are needed for our development are presented in Section 2. In Section 3 it is shown that the class of k-harmonic splines in $L_k^2(R^n)$ is a closed subspace of $L_k^2(R^n)$ whose corresponding orthogonal projection operator is quite natural; this is the key to what may be called the minimization property of these splines. Necessary and sufficient conditions on a sequence $\{v_{\mathbf{j}}\}$, $\mathbf{j}$ in Z^n, which allow it to be interpolated by the elements of $L_k^2(R^n)$, are given in Section 3; furthermore, it is shown that the unique element of minimal $L_k^2(R^n)$ norm which interpolates such a sequence is the k-harmonic spline interpolant.

The conventions and notation used here are identical to that in [1]. In particular, $SH_k(R^n)$ denotes the space of k-harmonic splines on R^n; k is always assumed to be an integer such that $2k \geqslant n+1$. The distributions L_k and Φ_k are defined by the formulas for their Fourier transforms,

$$\hat{L}_k(\xi) = (2\pi)^{-n/2} \frac{|\xi|^{-2k}}{\sum_{\mathbf{j} \in Z^n} |\xi - 2\pi\mathbf{j}|^{-2k}} \tag{2}$$

and

$$\hat{\Phi}_k(\xi) = |\xi|^{2k} \hat{L}_k(\xi). \tag{3}$$

Their properties which are relevant to our development are listed in [1]. Here we merely recall that L_k is called the fundamental function of interpolation; it is the unique k-harmonic spline such that $L_k(\mathbf{j}) = \delta_{0\mathbf{j}}$, $\mathbf{j}$ in Z^n, where $\delta_{0\mathbf{j}}$ is the Kronecker delta. In particular, every k-harmonic spline u enjoys the representation

$$u(x) = \sum_{\mathbf{j} \in Z^n} u(\mathbf{j})\, L_k(x - \mathbf{j}), \tag{4}$$

where the series converges absolutely and uniformly on compact subsets of R^n. For more details, background, and references see [1].

2. Definition and Properties of $L_k^2(R^n)$ and $l_k^2(Z^n)$

The linear space $L_k^2(R^n)$ is defined as the class of those tempered distributions u on R^n all of whose kth order derivatives are square integrable; in other words

$$L_k^2(R^n) = \{u \in \mathscr{S}'(R^n) \colon D^\nu u \text{ is in } L^2(R^n) \text{ for all } \nu \text{ with } |\nu| = k\}.$$

For this space a semi-inner product is given by

$$\langle u, v \rangle_k = \sum_{|\nu| = k} c_\nu \int_{R^n} D^\nu u(x)\, \overline{D^\nu v(x)}\, dx, \tag{5}$$

where the positive constants c_ν are specified by

$$|\xi|^{2k} = \sum_{|\nu| = k} c_\nu \xi^{2\nu}. \tag{6}$$

The semi-norm corresponding to (5) is denoted $\|u\|_{2,k}$, thus $\|u\|_{2,k}^2 = \langle u, u \rangle_k$. The null space of this seminorm is $\pi_{k-1}(R^n)$, the class of polynomials of degree less than or equal to $k - 1$.

Note that there are many seminorms equivalent to $\|u\|_{2,k}$ on $L_k^2(R^n)$. The reason for the particular choice used here is the fact that for u in $\mathscr{S}(R^n)$, by virtue of Plancherel's formula,

$$\|u\|_{2,k}^2 = \int_{R^n} |\xi|^{2k}\, |\hat{u}(\xi)|^2\, d\xi.$$

In view of (3) and (4), it is not difficult to conjecture that it is the minimization of this particular seminorm subject to the appropriate interpolatory conditions which leads to a solution which is a k-harmonic spline.

The objective of this section is to develop properties of $L_k^2(R^n)$ and its discrete analogue which is needed in our treatment of the variational problem in the following sections.

Since the norm on $L_k^2(R^n)$ only allows us to distinguish between the equivalence classes determined by it, dealing with the individual elements of $L_k^2(R^n)$ is rather slippery business. Nevertheless we regard the space $L_k^2(R^n)$ as a subspace of tempered distribution and not as a collection of equivalence classes. Thus if a distribution u is representable by a continuous function f, namely

$$\langle u, \phi \rangle = \langle f, \phi \rangle = \int_{R^n} f(x)\, \phi(x)\, dx$$

for all ϕ in $\mathscr{S}(R^n)$, then, following standard convention, we simply identify u with f and say that u is continuous. The following propositions allow us to get a somewhat better grip on the distributions in $L_k^2(R^n)$.

The first proposition follows from a routine argument and may be regarded as folklore.

PROPOSITION 1. *$\mathscr{S}(R^n)$ is dense in $L_k^2(R^n)$.*

The notion of a unisolvent set is needed in the statement of the next proposition. Recall that a *unisolvent set* Ω for $\pi_{k-1}(R^n)$ is a finite subset of R^n consisting of $k(n)$ elements with the property that if p is in $\pi_{k-1}(R^n)$ and $p(x) = 0$ for all x in Ω then $p(x) = 0$ for all x in R^n. Here $\pi_{k-1}(R^n)$ denotes the class of polynomials on R^n of degree no greater than $k-1$ and $k(n)$ denotes its dimension.

PROPOSITION 2. *Assume $2k \geqslant n+1$. Then the elements of $L_k^2(R^n)$ are continuous functions and there is a linear map P on $L_k^2(R^n)$ with the following properties:*

(i) *Pu is in $\pi_{k-1}(R^n)$.*

(ii) *$P^2u = Pu$.*

(iii) *If Ω is a unisolvent set for $\pi_{k-1}(R^n)$ then*

$$|Pu(x)| \leqslant C(1 + |x|^{k-1})(\|u\|_{2,k} + \|u\|_\Omega), \tag{7}$$

where $\|u\|_\Omega$ denotes the maximum of u on Ω and C is a constant which depends on Ω but is independent of u.

(iv) *If $Qu = u - Pu$ then Qu is continuous and satisfies*

$$\|Qu\|_{2,k} = \|u\|_{2,k} \tag{8}$$

and

$$|Qu(x)| \leqslant C(1 + |x|^k)\, \|u\|_{2,k}, \tag{9}$$

where C is a constant independent of u.

Proof. The operator P is defined by the formula for the Fourier transform of Pu. Namely, if ϕ is any element of $\mathscr{S}(R^n)$ then

$$\langle \widehat{Pu}, \phi \rangle = \langle \hat{u}, p_\phi \chi \rangle, \tag{10}$$

where p_ϕ is the Taylor polynomial of ϕ of degree $k-1$ and centered at 0, χ is a fixed function in $C_0^\infty(R^n)$ which is equal to one in a neighborhood of 0 and supported in the unit ball centered at 0. Of course $p_\phi \chi$ denotes the pointwise multiplication of p_ϕ and χ. (Note that the definition of P depends on the choice of χ.) Pu is well defined by formula (10). It is easy to check that P is linear and satisfies statements (i) and (ii) of the proposition. Next we verify (iv).

Write

$$Qu = u - Pu = u_1 + u_2, \tag{11}$$

where $\hat{u}_1 = \chi \widehat{Qu}$, $u_2 = Qu - u_1$, and χ is the function in the definition of P above. More specifically, for any ϕ in $\mathscr{S}(R^n)$

$$\langle \hat{u}_1, \phi \rangle = \langle \hat{u}, (\phi - p_\phi)\chi \rangle \tag{12}$$

and

$$\langle \hat{u}_2, \phi \rangle = \langle \hat{u}, (1-\chi)\phi \rangle. \tag{13}$$

Since $\hat{u}_1$ has compact support, u_1 is analytic and

$$u_1(x) = (2\pi)^{-n/2} \langle \hat{u}_1, e_x \rangle, \tag{14}$$

where $e_x(\xi)$ denotes the exponential $e^{i\langle x, \xi \rangle}$. Using (12) write

$$\langle \hat{u}_1, e_x \rangle = \langle \hat{u}, q_x \chi \rangle, \tag{15}$$

where

$$q_x(\xi)\, \chi(\xi) = [e_x(\xi) - p_{e_x}(\xi)]\, \chi(\xi) = \langle x, \xi \rangle^k \psi_x(\xi)\, \chi(\xi) \tag{16}$$

and ψ_x is analytic and bounded independent of x. Since

$$\langle x, \xi \rangle^k = \sum_{|\nu| = k} c_\nu x^\nu \xi^\nu$$

formulas (15) and (16) result in

$$\langle \hat{u}_1, e_x \rangle = \sum_{|\nu| = k} c_\nu x^\nu \langle \hat{u}_\nu, \psi_x \chi \rangle, \tag{17}$$

where $\hat{u}_\nu = \xi^\nu \hat{u}(\xi)$. Recalling the fact that if $|\nu| = k$ then $\hat{u}_\nu$ is in $L^2(R^n)$ with norm dominated by $\|u\|_{2,k}$ we see that (17) implies that

$$|\langle \hat{u}_1, e_x \rangle| \leqslant C\,|x|^k\,\|u\|_{2,k} \tag{18}$$

which together with (14) shows that

$$|u_1(x)| \leqslant C\,|x|^k\,\|u\|_{2,k}. \tag{19}$$

From (13) $\hat{u}_2 = (1-\chi)\hat{u}$ and thus $\hat{u}_2$ is in $L^1(R^n)$ since

$$\int_{R^n} (1-\chi(\xi))\,|\hat{u}(\xi)|\,d\xi \leqslant C \left\{ \int_{R^n} (1-\chi(\xi))^2\,|\xi|^{-2k}\,d\xi \right\}^{1/2} \|u\|_{2,k}.$$

This implies that u_2 is continuous and

$$|u_2(x)| \leqslant C\,\|u\|_{2,k}. \tag{20}$$

Now, u_1 and u_2 are both continuous and hence it follows that Qu is also. Identity (11) together with inequalities (19) and (20) imply (9). Since (8) is an immediate consequence of the fact that $u - Qu$ is in $\pi_{k-1}(R^n)$ the proof of statement (iv) is complete.

Finally, to see (iii) let Ω be the collection of points $\{x_1, ..., x_N\}$, where $N = k(n)$ is the dimension of $\pi_{k-1}(R^n)$, and let p_j, $j = 1, ..., N$ be the polynomials in $\pi_{k-1}(R^n)$ which are uniquely defined by $p_j(x_m) = \delta_{jm}$, where δ_{jm} is the Kronecker delta. Since $Pu = u - Qu$ is in $\pi_{k-1}(R^n)$ and Ω is unisolvent for $\pi_{k-1}(R^n)$ we see that

$$Pu(x) = \sum_{j=1}^{N} \{u(x_j) - Qu(x_j)\}\,p_j(x). \tag{21}$$

Now

$$|u(x_j) - Qu(x_j)| \leqslant |u(x_j)| + C(1 + |x_j|^k)\,\|u\|_{2,k} \tag{22}$$

and

$$|p_j(x)| \leqslant C_j(1 + |x|^{k-1}). \tag{23}$$

Formula (21) together with inequalities (22) and (23) imply the desired result. ▮

The operators P and Q are complementary orthogonal projections on $L_k^2(R^n)$. That is, every u in $L_k^2(R^n)$ can be expressed as

$$u = Pu + Qu. \tag{24}$$

It is clear from the proof that the operator P is not unique.

We now turn our attention to a discrete analogue of $L_k^2(R^n)$ which we refer to as $l_k^2(Z^n)$. This class of sequences may be described as follows.

First, recall that $\mathscr{Y}^\alpha$, α real, is the class of those sequences $u = \{u_\mathbf{j}\}$, $\mathbf{j}$ in Z^n, for which the norm

$$N_\alpha(u) = \sup_{\mathbf{j} \in Z^n} \frac{|u_\mathbf{j}|}{(1 + |\mathbf{j}|)^\alpha} \tag{25}$$

is finite. Let

$$\mathscr{Y}^\infty = \bigcup_\alpha \mathscr{Y}^\alpha \text{ and } \mathscr{Y}^{-\infty} = \bigcap_\alpha \mathscr{Y}^\alpha, \tag{26}$$

where the intersection and union are taken over all α, $-\infty < \alpha < \infty$. Also, we say that the sequence $u = \{u_\mathbf{j}\}$ is in $\pi_{k-1}(Z^n)$ if it is the restriction of an element p in $\pi_{k-1}(R^n)$ to Z^n; namely, $u_\mathbf{j} = p(\mathbf{j})$ for all $\mathbf{j}$ in Z^n, where p is in $\pi_{k-1}(Z^n)$.

Given a sequence u in $\mathscr{Y}^\infty$, for each i, $i = 1, ..., n$, $T_i u$ is the sequence

$$(T_i u)_\mathbf{j} = u_{\mathbf{j} + \mathbf{e}_i} - u_\mathbf{j},$$

where $\mathbf{e}_i$ is the n-tuple with 1 in the ith slot and 0 elsewhere. In other words, T_i is a difference operator in the direction $\mathbf{e}_i$. For any multi-index ν, T^ν is the usual composition (product) of $T_1, ..., T_n$, namely $T^\nu = T_1^{\nu_1} \cdots T_n^{\nu_n}$.

The space $l_k^2(Z^n)$ consists of those elements $u = \{u_\mathbf{j}\}$ in $\mathscr{Y}^\infty$ for which

$$\|u\|_{2,k}^2 = \sum_{\mathbf{j} \in Z^n} \sum_{|\nu| = k} |(T^\nu u)_\mathbf{j}|^2 \tag{27}$$

is finite. The corresponding semi-inner product is given by

$$\langle u, v \rangle_k = \sum_{|\nu| = k} \sum_{\mathbf{j} \in Z^n} (T^\nu u)_\mathbf{j} \overline{(T^\nu v)_\mathbf{j}}. \tag{28}$$

The notation used to denote discrete sequences and certain discrete sequence norms is identical to that used for the "analog" case considered earlier in this section. This should cause no confusion; the meaning should be clear from the context.

PROPOSITION 3. *$\mathscr{Y}^{-\infty}$ is dense in $l_k^2(Z^n)$.*

PROPOSITION 4. *There is a linear map P on $l_k^2(Z^n)$ with the following properties*:

(i) *Pu is in $\pi_{k-1}(Z^n)$.*

(ii) $P^2u = Pu$.

(iii) *If Ω is a unisolvent set for $\pi_{k-1}(Z^n)$ then*

$$|Pu_{\mathbf{j}}| \leqslant C(1 + |\mathbf{j}|^{k-1})(\|u\|_{2,k} + \|u\|_{\Omega}), \tag{29}$$

where $\|u\|_{\Omega}$ denotes the maximum of u on Ω and C is a constant which depends on Ω but is independent of u.

(iv) *If $Qu = u - Pu$*

$$\|Qu\|_{2,k} = \|u\|_{2,k} \tag{30}$$

and

$$|Qu_{\mathbf{j}}| \leqslant C(1 + |\mathbf{j}|^k)\, \|u\|_{2,k}, \tag{31}$$

where C is a constant independent of u.

The proof of these propositions essentially consists of identifying the space $l_k^2(Z^n)$ with an appropriate class of tempered distributions and applying the arguments used in proving the analogous facts for $L_k^2(R^n)$ in Propositions 1 and 2 *mutatis mutandis*. Indeed, observe that $\mathscr{Y}^{-\infty}$ equipped with the seminorms defined by (25) is a topological vector space whose dual can be identified with $\mathscr{Y}^{\infty}$. Now, the Fourier transform can be defined on $\mathscr{Y}^{-\infty}$ in the natural way; namely

$$\hat{u}(\xi) = (2\pi)^{-n/2} \sum_{\mathbf{j} \in Z^n} u(\mathbf{j}) e^{-i\langle \mathbf{j}, \xi \rangle}.$$

It maps $\mathscr{Y}^{-\infty}$ into the class of infinitely differentiable periodic functions on R^n. Hence the Fourier transform may be defined on $\mathscr{Y}^{\infty}$ via duality in the usual manner; it maps $\mathscr{Y}^{\infty}$ onto the class of periodic tempered distributions. With these identifications, it should be clear how to modify the arguments used in the proof of Propositions 1 and 2 so that they apply to $l_k^2(Z^n)$.

Another very useful identification may be described as follows.

Let $\mathscr{M}(Z^n)$ be the class of those tempered distributions f which enjoy the representation

$$f(x) = \sum_{\mathbf{j} \in Z^n} a_{\mathbf{j}}\, \delta(x - \mathbf{j}), \tag{32}$$

where $\delta(x)$ denotes the unit Dirac distribution at the origin. Note that $\mathscr{M}(Z^n)$ is a closed subspace of $\mathscr{S}'(R^n)$. Also recall that the Fourier transform is an isomorphism of $\mathscr{M}(Z^n)$ onto the class of periodic distributions.

Observe that $\mathscr{Y}^{\infty}$ and $\mathscr{M}(Z^n)$ are algebraically isomorphic via the mapping

$$u \to \sum u_{\mathbf{j}}\, \delta(x-\mathbf{j}). \tag{33}$$

Thus, whenever convenient, we may view elements of $\mathscr{Y}^{\infty}$ as being in $\mathscr{M}(Z^n)$ and vice versa.

Finite differences, such as those used in the definition of $l_k^2(Z^n)$, may be identified with elements of $\mathscr{M}(Z^n)$ as follows.

Let $\mathscr{F}$ be the subset of $\mathscr{M}(Z^n)$ consisting of those elements whose representation (32) contains at most a finite number of non-zero coefficients $a_{\mathbf{j}}$. The Fourier transform is an isomorphism of $\mathscr{F}$ onto the space $\mathscr{T}$ consisting of trigonometric polynomials. Note that $\mathscr{F}$ can be identified with a class of finite difference operators via convolution in the natural way; namely, if

$$T(x) = \sum a_{\mathbf{j}}\, \delta(x-\mathbf{j}) \tag{34}$$

is an element of $\mathscr{F}$ and u is any tempered distribution then

$$T * u(x) = \sum a_{\mathbf{j}} u(x-\mathbf{j})$$

is a finite difference of u. Thus we often refer to elements of $\mathscr{F}$ as finite differences.

In view of the identification (33), we may view $\mathscr{F}$ as the class of finite difference operators on $\mathscr{Y}^{\infty}$ or $\mathscr{S}'(R^n)$. In particular, the operators T^{ν} used in the definition of $l_k^2(Z^n)$ may be regarded as elements of $\mathscr{F}$ whose Fourier transforms are the trigonometric polynomials

$$\widehat{T^{\nu}} = (2\pi)^{-n/2}\, (e^{i\xi_1}-1)^{\nu_1} \cdots (e^{i\xi_n}-1)^{\nu_n}. \tag{35}$$

More generally, if T is a finite difference operator of form (34), we write Tu to denote $T * u$ if u is in $\mathscr{S}'(R^n)$ and, if u is in $\mathscr{Y}^{\infty}$, to denote the sequence representing the natural action of T on u, namely,

$$(Tu)_{\mathbf{j}} = \sum_{\mathbf{i}} a_{\mathbf{i}} u_{\mathbf{j}-\mathbf{i}}.$$

3. Polyharmonic Splines and $L_k^2(R^n)$

Recall that a continuous function f on R^n interpolates a sequence $u = \{u_{\mathbf{j}}\}$ if $f(\mathbf{j}) = u_{\mathbf{j}}$ for all $\mathbf{j}$ in Z^n. Proposition 2 together with existence and uniqueness for the cardinal interpolation problem for k-harmonic splines, see [1], clearly implies the following.

PROPOSITION 5. *If $2k \geqslant n+1$ and u is in $L_k^2(R^n)$ then u is continuous and of polynomial growth. The sequence of values $\{u(\mathbf{j})\}$, $\mathbf{j}$ in Z^n, is in $\mathscr{Y}^k$ and there is a unique k-harmonic spline which interpolates this sequence.*

If u is in $L_k^2(R^n)$ let $S_k u$ be the unique k-harmonic spline which interpolates the data sequence $\{u(\mathbf{j})\}$, $\mathbf{j}$ in Z^n; that is $S_k u(\mathbf{j}) = u(\mathbf{j})$ for all $\mathbf{j}$. Recall that we may write

$$S_k u(x) = \sum_{\mathbf{j} \in Z^n} u(\mathbf{j})\, L_k(x - \mathbf{j}), \tag{36}$$

where L_k is the fundamental spline defined in the Introduction. Clearly the mapping $u \to S_k u$ is linear and $S_k u = u$ whenever u is in $SH_k(R^n)$. In what follows we show that this mapping is an orthogonal projection of $L_k^2(R^n)$ onto $SH_k(R^n) \cap L_k^2(R^n)$.

Throughout the rest of this section P and Q denote a fixed pair of complementary projections whose existence is guaranteed by Proposition 2. We begin with some technical lemmas.

LEMMA 6. *Suppose $2k \geqslant n+1$ and $\{u_m\}$, $m = 1, \ldots,$ is a sequence in $L_k^2(R^n)$. If $\{u_m\}$ converges to u in $L_k^2(R^n)$ then*

(i) *$\{Qu_m\}$ converges to Qu uniformly on compact subsets of R^n.*

(ii) *$\{S_k Qu_m\}$ converges to $S_k Qu$ uniformly on compact subsets of R^n.*

Proof. Choose any positive numbers r and ε and observe that (i) follows if we can show that

$$|Qu_m(x) - Qu(x)| < \varepsilon \tag{37}$$

whenever m is sufficiently large and for all x such that $|x| < r$. From (9) we have

$$|Qu_m(x) - Qu(x)| \leqslant C(1 + |x|^k)\, \|u_m - u\|_{2,k}. \tag{38}$$

Now from (38) it is easy to see that choosing m so that

$$\|u_m - u\|_{2,k} < \varepsilon\{C(1 + r^k)\}^{-1}$$

implies (37).

To see (ii) observe that (36), (38), and the linearity of S_k imply

$$|S_k Qu_m(x) - S_k Qu(x)| \leqslant C\left\{\sum_{\mathbf{j} \in Z^n} (1 + |\mathbf{j}|^k)\, L_k(x - \mathbf{j})\right\} \|u_m - u\|_{2,k}.$$

Now, using the exponential decay of L_k, see [1], it is clear that (ii) follows from essentially the same reasoning as (i). ▮

LEMMA 7. *If* $2k \geqslant n+1$ *and* u *is in* $\mathscr{S}(R^n)$ *then* $S_k u$ *is in* $L_k^2(R^n)$ *and there is a constant* C, *independent of* u, *so that*

$$\|S_k u\|_{2,k} \leqslant C\, \|u\|_{2,k}. \tag{39}$$

Proof. Recall that

$$\widehat{S_k u(\xi)} = (2\pi)^{n/2}\, U(\xi)\, \hat{L}_k(\xi), \tag{40}$$

where, in view of Poisson's formula,

$$U(\xi) = (2\pi)^{-n/2} \sum_{\mathbf{j} \in Z^n} u(\mathbf{j}) e^{-i\langle \mathbf{j}, \xi\rangle} = \sum_{\mathbf{j} \in Z^n} \hat{u}(\xi - 2\pi \mathbf{j}).$$

Observe that by virtue of (36), (40), and Plancherel's formula, (39) is equivalent to

$$\int_{R^n} |\xi|^{2k}\, |U(\xi)\, \hat{L}_k(\xi)|^2\, d\xi \leqslant C \int_{R^n} |\xi|^{2k}\, |\hat{u}(\xi)|^2\, d\xi. \tag{41}$$

That $S_k u$ is in $L_k^2(R^n)$ follows from the readily transparent fact that the right-hand side of (41) is finite.

The remainder of this proof is devoted to demonstrating (41). This demonstration involves verifying the two inequalities

$$\int_{R^n} |\xi|^{2k}\, |U(\xi)\, \hat{L}_k(\xi)|^2\, d\xi \leqslant A \int_{Q^n} |\xi|^{2k}\, |U(\xi)|^2\, d\xi \tag{42}$$

and

$$\int_{Q^n} |\xi|^{2k}\, |U(\xi)|^2\, d\xi \leqslant B \int_{R^n} |\xi|^{2k}\, |\hat{u}(\xi)|^2\, d\xi, \tag{43}$$

where A and B are constants independent of u and Q^n is the cube

$$Q^n = \{\xi\colon -\pi < \xi_j \leqslant \pi,\ j = 1, ..., n\}.$$

It should be clear that (42) and (43) imply (41).

To see (42) recall that $|\xi|^{2k}\, |\hat{L}_k(\xi)|^2 = \hat{\Phi}_k(\xi)\, \hat{L}_k(\xi)$, where $\hat{\Phi}_k$ is the periodic function defined by (3); if necessary, consult [1] for more details concerning this function. Let $Q^n_{\mathbf{j}} = 2\pi\mathbf{j} + Q^n$ and write

$$\int_{R^n} |\xi|^{2k} \, |U(\xi)\, \hat{L}_k(\xi)|^2 \, d\xi = \sum_{\mathbf{j}\in Z^n} \int_{Q^n_{\mathbf{j}}} |U(\xi)|^2 \, \hat{\Phi}_k(\xi)\, \hat{L}_k(\xi)\, d\xi$$

$$= \sum_{\mathbf{j}\in Z^n} \int_{Q^n} |U(\xi)|^2 \, \hat{\Phi}_k(\xi)\, \hat{L}_k(\xi + 2\pi\mathbf{j})\, d\xi, \quad (44)$$

where the last equality follows from a change of variable of integration and the periodicity of U and $\hat{\Phi}_k$. Now let $a_{\mathbf{j}}$ be the maximum of $\hat{L}_k(\xi + 2\pi\mathbf{j})$ for ξ in Q^n and note that $0 \leqslant \hat{\Phi}_k(\xi) = |\xi|^{2k}\, \hat{L}_k(\xi) \leqslant a_0\, |\xi|^{2k}$ if ξ is in Q^n. Since $a_{\mathbf{j}} = O(|\mathbf{j}|^{-2k})$ for large $\mathbf{j}$, we may write

$$\sum_{\mathbf{j}\in Z^n} \int_{Q^n} |U(\xi)|^2 \, \hat{\Phi}_k(x)\, \hat{L}_k(\xi + 2\pi\mathbf{j})\, dx \leqslant a_0 \left(\sum_{\mathbf{j}\in Z^n} a_{\mathbf{j}} \right) \int_{Q^n} |\xi|^{2k}\, |U(\xi)|^2 \, dx. \quad (45)$$

Formula (44) and inequality (45) imply (42) with $A = a_0(\sum_{\mathbf{j}\in Z^n} a_{\mathbf{j}})$.

To see (43) observe that for ξ in Q^n we may write

$$|\xi|^{2k}\, |U(\xi)|^2 \leqslant \left(|\xi|^k \sum_{\mathbf{j}\in Z^n} |\hat{u}(\xi - 2\pi\mathbf{j})| \right)^2$$

$$\leqslant \left(\sum_{\mathbf{j}\in Z^n} b_{\mathbf{j}}\, |\xi - 2\pi\mathbf{j}|^k \, |\hat{u}(\xi - 2\pi\mathbf{j})| \right)^2$$

$$\leqslant \sum_{\mathbf{i}\in Z^n} \sum_{\mathbf{j}\in Z^n} b_{\mathbf{i}}\, |\xi - 2\pi\mathbf{i}|^k \, |\hat{u}(\xi - 2\pi\mathbf{i})| \times b_{\mathbf{j}}\, |\xi - 2\pi\mathbf{j}|^k \, |\hat{u}(\xi - 2\pi\mathbf{j})|,$$

where $b_0 = 1$ and otherwise $b_{\mathbf{j}}$ is equal to the maximum of $|\xi - 2\pi\mathbf{j}|^{-k}$ over ξ in Q^n. Integrating the last expression involving U over Q^n and observing that

$$\int_{Q^n} |\xi - 2\pi\mathbf{i}|^k \, |\hat{u}(\xi - 2\pi\mathbf{i})| \; |\xi - 2\pi\mathbf{j}|^k \, |\hat{u}(\xi - 2\pi\mathbf{j})| \, d\xi \leqslant V_{\mathbf{i}} V_{\mathbf{j}},$$

where

$$V_{\mathbf{j}} = \left(\int_{Q^n} |\xi - 2\pi\mathbf{j}|^{2k} \, |\hat{u}(\xi - 2\pi\mathbf{j})|^2 \, d\xi \right)^{1/2}$$

allows us to write

$$\int_{Q^n} |\xi|^{2k}\, |U(\xi)|^2 \, d\xi \leqslant \sum_{\mathbf{i}\in Z^n} \sum_{\mathbf{j}\in Z^n} b_{\mathbf{i}} b_{\mathbf{j}} V_{\mathbf{i}} V_{\mathbf{j}} = \left(\sum_{\mathbf{j}\in Z^n} b_{\mathbf{j}} V_{\mathbf{j}} \right)^2. \quad (46)$$

Note that $2k>n$ implies that the sum $\sum_{\mathbf{j}\in Z^n} b_{\mathbf{j}}^2$ is finite; thus by virtue of (46) and Schwarz's inequality we have

$$\int_{Q^n} |\xi|^{2k} |U(\xi)|^2 \, d\xi \leqslant \Big(\sum_{\mathbf{j}\in Z^n} b_{\mathbf{j}}^2 \Big)\Big(\sum_{\mathbf{j}\in Z^n} V_{\mathbf{j}}^2 \Big). \tag{47}$$

Since

$$\sum_{\mathbf{j}\in Z^n} V_{\mathbf{j}}^2 = \int_{R^n} |\xi|^{2k} |\hat{u}(\xi)|^2 \, d\xi,$$

(47) implies (43) with $B=\sum b_{\mathbf{j}}^2$. ∎

PROPOSITION 8. *Suppose* $2k \geqslant n+1$ *and* u *is in* $L_k^2(R^n)$. *Then* $S_k u$ *is* $L_k^2(R^n)$ *and there is a constant* C *independent of* u *so that*

$$\|S_k u\|_{2,k} \leqslant C \, \|u\|_{2,k}. \tag{48}$$

Proof. By Lemma 7, S_k maps the dense subspace $\mathscr{S}(R^n)$ of $L_k^2(R^n)$ continuously into $L_k^2(R^n)$. Let $\tilde{S}_k$ denote the continuous extension of S_k onto all of $L_k^2(R^n)$. The proposition follows if we can show that

$$S_k u = \tilde{S}_k u \tag{49}$$

in $L_k^2(R^n)$ for all u in $L_k^2(R^n)$.

To see (49) let P and Q be a pair of operators whose existence is guaranteed by Proposition 2. Now let u be any element of $L_k^2(R^n)$ and let $\{u_m\}$ be a sequence in $\mathscr{S}(R^n)$ converging to u in $L_k^2(R^n)$. By virtue of Lemma 6 the sequences $\{QS_k u_m\}$ and $\{S_k Q u_m\}$ convergence to $Q\tilde{S}_k u$ and $S_k Q u$, respectively, uniformly on compact subsets of R^n. Since $\|QS_k u_m - S_k Q u_m\|_{2,k}=0$ it follows that $Q\tilde{S}_k u - S_k Q u = p$, where p is a polynomial in $\pi_{k-1}(R^n)$. Hence

$$\tilde{S}_k u - S_k u = q, \tag{50}$$

where q is a polynomial in $\pi_{k-1}(R^n)$. From (50) it follows that $S_k u$ is in $L_k^2(R^n)$ and satisfies (49). ∎

THEOREM 9. *The mapping* $u \to S_k u$ *is an orthogonal projection of* $L_k^2(R^n)$ *onto* $SH_k(R^n) \cap L_k^2(R^n)$.

Proof. In view of Proposition 8 it suffices to show that S_k is idempotent and self-adjoint.

Since $S_k u$ is a k-harmonic cardinal spline for any u in $L_k^2(R^n)$ and, by virtue of the uniqueness of cardinal interpolation, $S_k u = u$ for all u in

$SH_k(R^n) \cap L_k^2(R^n)$ it follows that $S_k(S_k u) = S_k u$ for all u in $L_k^2(R^n)$ and hence S_k is idempotent.

To see that S_k is self-adjoint let u and v be any elements of $\mathscr{S}(R^n)$ and, as in the proof of Lemma 7, let U and V be the periodizations of $\hat{u}$ and $\hat{v}$, namely,

$$U(\xi) = \sum_{\mathbf{j} \in Z^n} \hat{u}(\xi - 2\pi \mathbf{j})$$

and a similar formula for V. Recall that $\widehat{S_k u} = (2\pi)^{n/2}\, U\hat{L}_k$ and $|\xi|^{2k}\, \hat{L}_k(\xi) = \hat{\Phi}_k(\xi)$, use Plancherel's formula and the fact that U, V, and $\hat{\Phi}_k$ are periodic, and write

$$\begin{aligned}
(2\pi)^{-n/2} \langle S_k u, v \rangle_k &= \int_{R^n} |\xi|^{2k}\, U(\xi)\, \hat{L}_k(\xi)\, \overline{\hat{v}(\xi)}\, d\xi \\
&= \int_{R^n} U(\xi)\, \hat{\Phi}_k(\xi)\, \overline{\hat{v}(\xi)}\, d\xi \\
&= \int_{Q^n} U(\xi)\, \overline{\hat{\Phi}_k(\xi)\, V(\xi)}\, d\xi \\
&= \int_{R^n} \hat{u}(\xi)\, \overline{\hat{\Phi}_k(\xi)\, V(\xi)}\, d\xi \\
&= \int_{R^n} |\xi|^{2k}\, \hat{u}(\xi)\, \overline{V(\xi)\, \hat{L}_k(\xi)}\, d\xi \\
&= (2\pi)^{-n/2} \langle u, S_k v \rangle_k,
\end{aligned}$$

where Q^n is the cube $\{\xi\colon \pi < \xi_j \leqslant \pi,\ j = 1, ..., n\}$. Hence

$$\langle S_k u, v \rangle_k = \langle u, S_k v \rangle_k \tag{51}$$

holds for u and v in $\mathscr{S}(R^n)$. Since $\mathscr{S}(R^n)$ is dense in $L_k^2(R^n)$ and S_k is continuous it follows that (51) holds for all u and v in $L_k^2(R^n)$ and thus S_k is self-adjoint. ▌

Theorem 9 together with elementary facts concerning orthogonal projections on Hilbert spaces imply some interesting facts concerning k-harmonic splines in this class. We list several transparent corollaries.

PROPOSITION 10. *If* $2k \geqslant n+1$ *then* $SH_k(R^n) \cap L_k^2(R^n)$ *is a closed subspace of* $L_k^2(R^n)$.

PROPOSITION 11. *If* $2k \geqslant n+1$ *then the following holds for all* u *in* $L_k^2(R^n)$:

$$\|u\|_{2,k} = \|u - S_k u\|_{2,k} + \|S_k u\|_{2,k} \tag{52}$$

and thus

$$\|S_k u\|_{2,k} \leqslant \|u\|_{2,k}. \tag{53}$$

Suppose u is any element in $L_k^2(R^n)$ and consider the sequence of values $\{u(\mathbf{j})\}$, $\mathbf{j}$ in Z^n. We define M_u to be that subset of $L_k^2(R^n)$ consisting of those elements v such that $v(\mathbf{j}) = u(\mathbf{j})$ for all $\mathbf{j}$ in Z^n. Clearly M_u is an affine subspace of $L_k^2(R^n)$. In view of (53) it is not difficult to see the following concerning M_u and S_k.

PROPOSITION 12. *If $2k \geqslant n+1$ then there exists a unique element w in M_u such that*

$$\|w\|_{2,k} = \min_{v \in M_u} \|v\|_{2,k},$$

and $w = S_k u$. In other words, $S_k u$ is the unique element in M_u of minimal $L_k^2(R^n)$ norm.

4. CARDINAL INTERPOLATION IN $L_k^2(R^n)$ AND $l_k^2(Z^n)$

As mentioned in the introduction, in this section we present necessary and sufficient conditions on a sequence $\{v_{\mathbf{j}}\}$, $\mathbf{j}$ in Z^n, which allow it to be interpolated by the elements of $L_k^2(R^n)$. In addition the interpolating element of minimal $L_k^2(R^n)$ norm is characterized.

First recall the definitions of the class $\mathscr{F}$ of finite difference operators given earlier. We say that T in $\mathscr{F}$ is of order k if $\hat{T}(\xi) = O(|\xi|^k)$ but not $o(|\xi|^k)$ as ξ goes to 0.

PROPOSITION 13. *Suppose $2k \geqslant n+1$, u is in $L_k^2(R^n)$, and T is any finite difference operator of order $\geqslant k$. Then*

$$\sum_{\mathbf{j} \in Z^n} |Tu(j)|^2 \leqslant C \|u\|_{2,k}^2, \tag{54}$$

where C is a constant independent of u.

Proof. If u is in $\mathscr{S}(R^n)$, let

$$U(\xi) = (2\pi)^{-n/2} \sum_{\mathbf{j} \in Z^n} u(j) e^{-i\langle \mathbf{j}, \xi \rangle}$$

and recall the definition of $\Phi_k(\xi)$. By virtue of Parseval's identity and the fact that $|\hat{T}(\xi)|^2/\hat{\Phi}_k(\xi)$ is bounded we may write

$$\begin{aligned}
\sum_{\mathbf{j}\in Z^n} |Tu(\mathbf{j})|^2 &= \int_{Q^n} |\hat{T}(\xi)\, U(\xi)|^2\, d\xi \\
&= \int_{Q^n} |U(\xi)|^2\, |\hat{T}(\xi)|^2\, |\hat{\Phi}_k(\xi)|^2\, |\hat{\Phi}_k(\xi)|^{-2}\, d\xi \\
&\leqslant C \int_{Q^n} |U(\xi)|^2\, |\hat{\Phi}_k(\xi)|^2\, |\hat{\Phi}_k(\xi)|^{-1}\, d\xi \\
&= C \int_{R^n} |U(\xi)\, \hat{\Phi}_k(\xi)|^2\, |\xi|^{-2k}\, d\xi \\
&= C \int_{R^n} |U(\xi)\, \hat{\Phi}_k(\xi)|\, \xi|^{-2k}|^2\, |\xi|^{2k}\, d\xi \\
&= C\, \|S_k u\|_{2,k}.
\end{aligned}$$

Since $\|S_k u\|_{2,k} \leqslant \|u\|_{2,k}$ we may conclude that (54) holds whenever u is in $\mathscr{S}(R^n)$. To see that (54) holds for all u in $L_k^2(R^n)$ let P and Q be the operators whose existence is guaranteed by Proposition 2, recall that $Tp=0$ for all polynomials in $\pi_{k-1}(R^n)$, and observe that

$$TQu = Tu \tag{55}$$

for all u in $L_k^2(R^n)$. Choose any element u in $L_k^2(R^n)$, and let $\{u_m\}$ be a sequence in $\mathscr{S}(R^n)$ converging to u in $L_k^2(R^n)$. By virtue of Lemma 6 $\{Qu_m\}$ converges to Qu uniformly on compact subsets of R^n and hence in view of (55) $\{Tu_m(\mathbf{j})\}$ converges to $Tu(\mathbf{j})$ for all $\mathbf{j}$ in Z^n. This last observation together with the fact that (54) holds for each u_m implies that

$$\sum |Tu(\mathbf{j})|^2 \leqslant C\, \|u\|_{2,k}^2,$$

where the sum is taken over any finite subset of Z^n. The last inequality of course implies the desired result. ∎

We say that a finite collection T_m, $m=1, ..., N$, of finite differences satisfies condition τ_k if

$$\hat{\Phi}_k(\xi) \left(\sum_{m=1}^{N} |\hat{T}_m(\xi)|^2 \right)^{-1}$$

is in $L^\infty(R^n)$. Note that the T^ν's used in the definition of $l_k^2(Z^n)$ enjoy this property. Also note that for every positive integer k there is a finite collection in $\mathscr{F}$ which satisfies condition τ_k.

Proposition 14. *Suppose $2k \geqslant n+1$, u is a sequence in $l_k^2(Z^n)$, and f_u is* *the unique k-harmonic cardinal spline which interpolates u. Then* f_u *is in* $L_k^2(R^n)$ *and*

$$\|f_u\|_{2,k} \leqslant C\,\|u\|_{2,k}, \tag{56}$$

where C is a constant independent of u.

Proof. Suppose u is in $\mathscr{Y}^{-\infty}$; then

$$f_u = \sum_{\mathbf{j} \in Z^n} u_{\mathbf{j}} L_k(x - \mathbf{j}),$$

the function

$$U(\xi) = \sum_{\mathbf{j} \in Z^n} u_{\mathbf{j}} e^{-i\langle \mathbf{j}, \xi \rangle}$$

is well defined, and $\widehat{f_u} = U\hat{L}_k$. Applying Plancherel's formula and the fact that the collection $\{T^\nu\colon |\nu| = k\}$ satisfies condition τ_k we may write

$$\begin{aligned}
\|f_u\|_{2,k}^2 &= \int_{R^n} |U(\xi)\,\hat{L}_k(\xi)|^2\,|\xi|^{2k}\,d\xi \\
&= \int_{R^n} |U(\xi)|^2\,\hat{\Phi}_k^2(\xi)\,\hat{L}_k(\xi)\,d\xi \\
&= \int_{Q^n} |U(\xi)|^2\,\hat{\Phi}_k(\xi)\,d\xi \\
&= \int_{Q^n} |U(\xi)|^2 \left\{\sum_{\nu = k} |\widehat{T^\nu}(\xi)|^2\right\}\left\{\frac{\hat{\Phi}_k(\xi)}{\sum_{|\nu| = k} |\widehat{T^\nu}(\xi)|^2}\right\} d\xi \\
&\leqslant C \int_{Q^n} \sum_{|\nu| = k} |\widehat{T^\nu}(\xi)\,U(\xi)|^2\,d\xi \\
&= C\,\|u\|_{2,k}^2.
\end{aligned}$$

Thus (56) holds for u in $\mathscr{Y}^{-\infty}$. In view of Propositions 3 and 4 the desired result follows from a density argument similar to that use in the proof of Proposition 13. ∎

COROLLARY 1. *The mapping*

$$\{u_{\mathbf{j}}\} \to f_u = \sum_{\mathbf{j} \in Z^n} u_{\mathbf{j}}\, L_k(x - \mathbf{j})$$

is an isomorphism between $l_k^2(Z^n)$ and $L_k^2(R^n) \cap SH_k(R^n)$ such that

$$c \|u\|_{2,k} \leqslant \|f_u\|_{2,k} \leqslant C \|u\|_{2,k},$$

where c and C are positive constants independent of u.

We conclude this paper by summarizing the contents of the above two propositions in the following theorems.

THEOREM 15. *Given a sequence $u = \{u_{\mathrm{j}}\}$, there is an element in $L_k^2(R^n)$ which interpolates it if and only if u is in $l_k^2(Z^n)$.*

Combining the results in this section together with Propositions 11 and 12 easily produces the following conclusion.

THEOREM 16. *Suppose u is in $l_k^2(Z^n)$. Then there is a unique k-harmonic spline f_u in $L_k^2(R^n) \cap SH_k(R^n)$ which interpolates u. This interpolant f_u has the property that*

$$\|f_u\|_{2,k} < \|g\|_{2,k}$$

for any g in $L_k^2(R^n)$ which interpolates u, unless $g(x) = f_u(x)$ for all x in R^n. In other words, f_u is the unique element of minimal $L_k^2(R^n)$ norm which interpolates u.

REFERENCES

1. W. R. MADYCH AND S. A. NELSON, Polyharmonic cardinal splines, *J. Approx. Theory* **60** (1990), 141–156.
2. I. J. SCHOENBERG, "Cardinal Spline Interpolation," CBMS Vol. 12, SIAM, Philadelphia, PA, 1973.